FLATNESS AND LQR CONTROL OF FURUTA PENDULUM

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Abstract: This paper presents comparison of two control algorithms, Flatness and LQR, which are used to control rotational inverted pendulum, a highly nonlinear and unstable system. Flatness is very effective for nonlinear systems in trajectory tracking tasks. LQR is one of the optimal control techniques, which takes into account the states of the dynamical system and control input to make optimal control decisions. Both algorithms are implemented on the real system under laboratory conditions and obtained results show a comparative advantage of Flatness algorithm in tracking the reference trajectory.

1. INTRODUCTION

Rotational inverted pendulum or Furuta pendulum, consist of a driven disk which rotates in the horizontal plane and a pendulum attached to that disk which is free to rotate in the vertical plane. It was invented in 1992 at Tokyo Institute of Technology by Katsuhisa Furuta and his colleagues [1]. It is an example of a complex nonlinear system. The pendulum is "underactuated" mechanical system i.e. it has lower number of actuators than degrees of freedom. In recent years it draws the attention of the control community as a platform for the development of various classical and modern control laws.

This paper describes comparative analysis of two control algorithms for the pendulum: LQR and flatness control. LQR (linear quadratic regulator) is a widely used method in optimal control of linear systems. It operates a dynamic system at minimum cost. On the other hand, differential flatness control is very effective for nonlinear system [2]. Flatness based control has been developed and applied in many industrial processes with great success in planning and tracking reference trajectory e.g. in thermal and chemical processes [3][4], electric drives [5], etc. Flat system is a control system described with vector of variables called "flat output", which can be used to explicitly express all states and inputs in terms of the flat output and finite number of its derivatives. Thus, finding the flat output, system dynamics is parameterized.

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Both algorithms have been implemented in the laboratory and experiment results have been shown for the case of disk rotation for a predefined angle (trajectory tracking), while stabilizing the pendulum in an upright (inverse) position.

2. MATHEMATICAL MODEL OF FURUTA PENDULUM

Fig. 1. shows Furuta pendulum sketch with the parameters necessary to describe system dynamics using differential equations. The angular displacements of the disk and pendulum are θ_1 and θ_2, respectively. Pendulum consists of a rod of mass m_r and weight of mass m_w. By moving the weight along the rod, pendulum center of gravity is changed. Length of rod is L_r, while the distances from the hub (disk-pendulum joint) to the weight and end of the rod are y_m and y_r, respectively. Distance from disk center to the pendulum is R_h while J_1 represents moment of inertia of the disk and all the components that uniformly rotate together with the disk. The viscous friction coefficient is denoted by c_1.

Dynamic model described with (1) and (2) is carried out using Euler-Lagrange equations of motions for mechanical systems [6]. Under assumption that control will be performed for small displacements of the disk and pendulum, the system is linearized near the equilibrium point ($\theta_1 = 0$ and $\theta_2 = 0$) and yields:

$$\ddot{\theta}_1\left[\overline{J}_1 + J_y\right] + \ddot{\theta}_2 m R_h l_{cg} + c_1\dot{\theta}_1 = T \tag{1}$$

$$\ddot{\theta}_1 m R_h l_{cg} + \ddot{\theta}_2 \overline{J}_z - mgl_{cg}\theta_2 = 0 \tag{2}$$

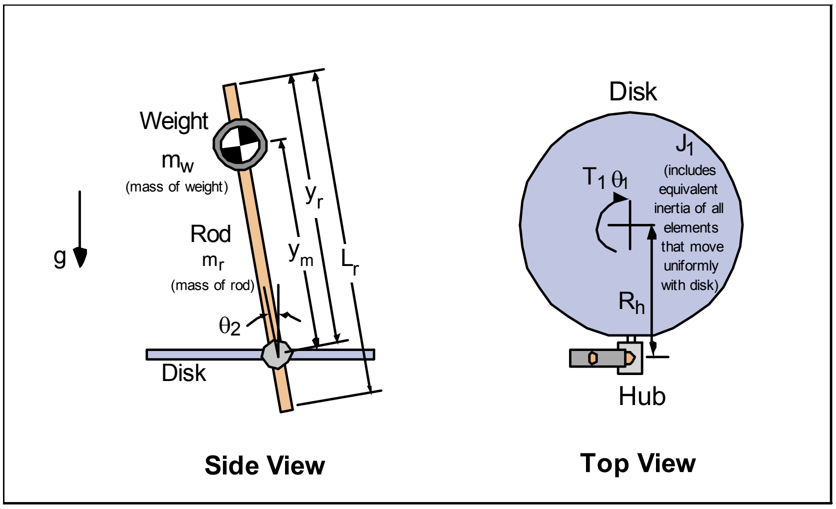

Fig. 1. Furuta pendulum sketch

Auxiliary variables for the model are derived as: $m = m_r + m_w$, $l_r = y_r - L_r / 2$, $l_w = y_m$, $l_{cg} = (m_r l_r + m_w l_w) / m$, $\bar{J}_1 = J_1 + mR_h^2$, $J_y = \frac{1}{4} m_w R_w^2$, $\bar{J}_z = \frac{1}{12} m_r L_r^2 + \frac{1}{2} m_w R_w^2 + m l_{cg}^2$. Also, mathematical model includes dynamics of other components such as DC motor, drive pulley, digital-analog converter (DAC), servo amplifier and optical encoders. However, dynamics of these components is much faster than the mechanical pendulum system so it will be described only with gain k_s.

For the sake of LQR design, equations (1) and (2) may be written in state space model as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}T \tag{3}$$

where:

$$\mathbf{x} = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad \mathbf{A} = \frac{1}{p} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -m^2 l_{cg}^2 R_h g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & c_1 m R_h l_{cg} & m l_{cg} (\bar{J}_1 + J_y) & 0 \end{bmatrix}$$
$$\mathbf{b} = \frac{1}{p} \begin{bmatrix} 0 \\ \bar{J}_z \\ 0 \\ -m R_h l_{cg} \end{bmatrix} \quad p = \bar{J}_z (\bar{J}_1 + J_y) - (m R_h l_{cg})^2 \tag{4}$$

3. FLATNESS CONTROL

Mathematical model of Furuta pendulum consists of two independent equations, (1) and (2). At the same time, the system is described by three variables (θ_1, θ_2, T). By definition, the number of the flat outputs is determined by subtracting the number of the equations from the number of the variables, giving one flat output for pendulum. This flat output is linear combination of displacements θ_1 and θ_2:

$$y = k_1 \theta_1 + k_2 \theta_2 \tag{5}$$

Introducing $k_1 = R_h l_{cg} / g$ and $k_2 = \bar{J}_z / m g l_{cg}$, from (2) we can express θ_2 as:

$$\theta_2 = \ddot{y} \tag{6}$$

and from (5) θ_1 as:

$$\theta_1 = -\frac{1}{k_1}(k_2\theta_2 + y) = -\frac{1}{k_1}(k_2\ddot{y} + y) \tag{7}$$

whereas motor torque is obtained by combining (6) and (7) with (1):

$$T = -\frac{\bar{J}_z}{mR_h l_{cg}^2}(\bar{J}_1 + J_y + mR_h l_{cg})\ddddot{y} - \frac{c_1\bar{J}_z}{mR_h l_{cg}^2}\dddot{y} - \frac{(\bar{J}_1 + J_y)g}{R_h l_{cg}}\ddot{y} - \frac{c_1 g}{R_h l_{cg}}\dot{y} = f(\ddddot{y}, \dddot{y}, \ddot{y}, \dot{y}) \tag{8}$$

Hence, all system variables are expressed in terms of the flat output and its derivatives as we can see from (6)-(8). Now it is possible to design feed-forward control by choosing reference trajectory of the flat output $y(t) = y_r(t)$, so that all the reference trajectories of other variables ($\theta_{1r}, \theta_{2r}, T_r$) will be defined by the aforementioned relations.

Choice of the reference trajectory is problem known as "motion planning". It is necessary for the trajectory to satisfy initial and final conditions i.e. we must define initial (at $t = 0$) and final (at $t = t_f$) position of disk and pendulum as well as their angular velocities and accelerations. The main control task is to disk follow the given trajectory for the specified angle θ_{1f} and time t_f with no oscillations at the beginning and end of the motion. This is achieved by setting all derivatives of θ_1 and θ_2 equal to zero. These conditions are written as:

$$\begin{aligned}
&\theta_1(0) = 0, \ \ \theta_1(t_f) = \theta_{1f}, \quad \theta_2(0) = 0, \quad \theta_2(t_f) = 0, \\
&\dot{\theta}_1(0) = \dot{\theta}_1(t_f) = 0, \qquad \dot{\theta}_2(0) = \dot{\theta}_2(t_f) = 0, \\
&\ddot{\theta}_1(0) = \ddot{\theta}_1(t_f) = 0, \qquad \ddot{\theta}_2(0) = \ddot{\theta}_2(t_f) = 0.
\end{aligned} \tag{9}$$

From (8) we can see that torque contains derivatives of the flat output up to the fourth order resulting the four times differentiable trajectory for the flat output. In addition, this trajectory must satisfy conditions (9) and the best way to describe it analytically is using polynomials. The required trajectory is 9th order polynomial:

$$y_r(t) = \sum_{k=0}^{9} c_k t^k \tag{10}$$

where unknown coefficients $c_0,...,c_9$ are calculated combining (5) and (6) with (9) and differentiating (10).

Feed-forward control is designed under the following assumptions: mathematical model coincides with the physical system, there are no external disturbances and initial conditions meet the actual. In practice this is never the case and system is unstable, so to stabilize the pendulum, designing a feedback will be the next task. In this paper, feedback system is designed by defining a tracking error:

$$e(t) = y(t) - y_r(t) \tag{11}$$

Tracking error dynamics is linear and for the given system takes form of the fourth-order differential equation:

$$\ddddot{e} + k_3\dddot{e} + k_2\ddot{e} + k_1\dot{e} + k_0 e = 0 \tag{12}$$

and can be rephrased as:

$$\ddddot{y} = \ddddot{y}_r - K_3(\dddot{y} - \dddot{y}_r) - K_2(\ddot{y} - \ddot{y}_r) - K_1(\dot{y} - \dot{y}_r) - K_0(y - y_r) \tag{13}$$

The coefficients K_0, K_1, K_2, K_3 from (13) are chosen to meet the convergence requirements towards the reference trajectories. They are chosen so that tracking error poles are in the complex right half-plane. The reference trajectories of the flat output on the right side of (13) are obtained by differentiating (10), whereas actual values of the flat output are obtained from (5) and (6) as:

$$y = k_1\theta_1 + k_2\theta_2 \tag{14}$$

$$\dot{y} = k_1\dot{\theta}_1 + k_2\dot{\theta}_2 \tag{15}$$

$$\ddot{y} = \theta_2 \tag{16}$$

$$\dddot{y} = \dot{\theta}_2 \tag{17}$$

which implies measuring the position and velocity of the disk and pendulum ($\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$). Then, (14)-(17) with (13) are used to calculate the drive torque of the motor according to (8). Feedback control is shown in Figure 3.

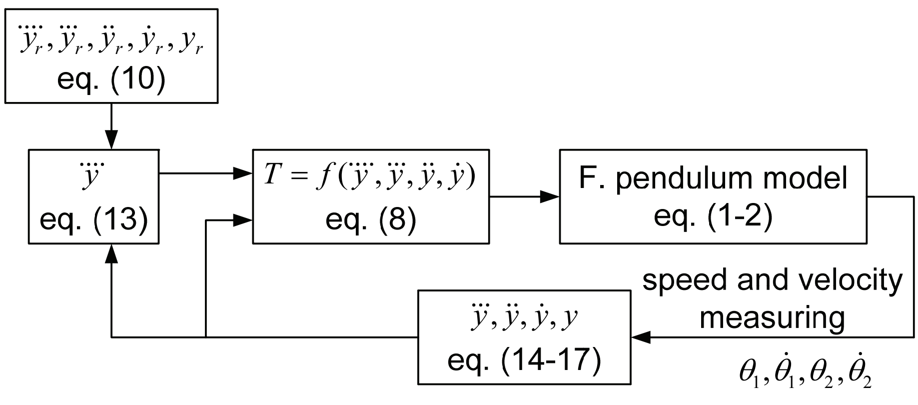

Fig. 2. Feedback control system

4. LQR CONTROL

Optimal control theory aims to minimize the systems dynamics according to selected criteria. Furuta pendulum state space model is described in (4). Cost function (performance index) for minimization is:

$$J = \int_0^\infty (\mathbf{x}^T \mathbf{Q}\mathbf{x} + \mathbf{u}^T \mathbf{R}\mathbf{u})dt \tag{18}$$

In this way disk position error and control effort are minimized. By adjusting the weight matrices ***Q*** and ***R*** we compromise between the system response and input energy. Optimal control is state feedback control:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \tag{19}$$

where

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \tag{20}$$

and matrix ***P*** is solution of Riccati equation:

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = 0 \tag{21}$$

5. EXPERIMENT RESULTS

The laboratory model of Furuta pendulum - ECP A51 [7], consists of three subsystems (Fig. 3.): electromechanical (Furuta pendulum, DC motor that drives the disk and encoders that measure the angular position of the disk and pendulum), electronic (Real-Time controller) and software (executive program). Control task is finding a suitable control law for pendulum stabilization in inverted position.

Fig. 3. Furuta pendulum lab model

The parameter values necessary for pendulum model are given in Table 1.

Table 1. Experiment parameters

Symbol	Value	Unit
m_r	0.069	kg
m_w	0.089	kg
L_r	0.43	m
y_m	0.32	m
y_r	0.42	m
R_h	0.25	m
J_1	0.0166	kgm^2
c_1	0.02	kgm^2/s
k_s	0.53	Nm/rad

In both control laws, the experiment was done for disk rotation by 30° for 2 seconds with tracking θ_{1r} trajectory. After the rotation, disk drive needs to stabilize the pendulum for additional 2 seconds. For the flatness control, coefficients in (12) or (13) was chosen to ensure error convergence fast enough. The error dynamics contains 4 right-half plane poles at -12, from where $(K_0, K_1, K_2, K_3) = (20736, 6912, 864, 48)$. For LQR control the selected values are: $\boldsymbol{Q}$=$\boldsymbol{C}^T\boldsymbol{C}$, $\boldsymbol{C}$=[1 0 0 0], so that $\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$=$\theta_1$, which results in minimization of disk position error. The value of the weight matrix $\boldsymbol{R}$=r is selected so that the bandwidth of the closed loop system is as large as possible, and that the frequency of the poles is less than practical limit for this system, 3 Hz [7]. This is satisfied for r = 10. Using the (20) and (21), with chosen $\boldsymbol{Q}$ i $\boldsymbol{R}$, it leads us to solutions for $\boldsymbol{K}$=[-0.316 -0.139 -1.211 -0.211].

Figures 3, 4. and 5. show experimental results for a given displacement of the disc.

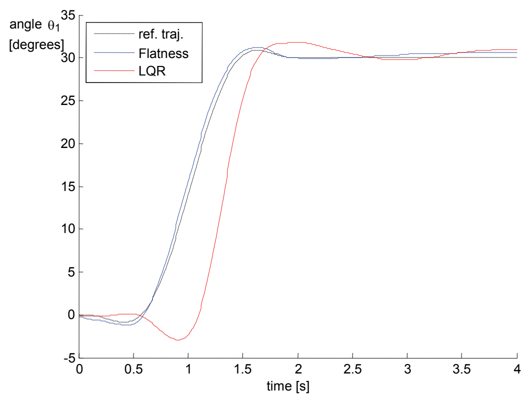

Fig. 3. Trajectory tracking of θ_{1r} (dashed) using flatness (blue) and LQR (red)

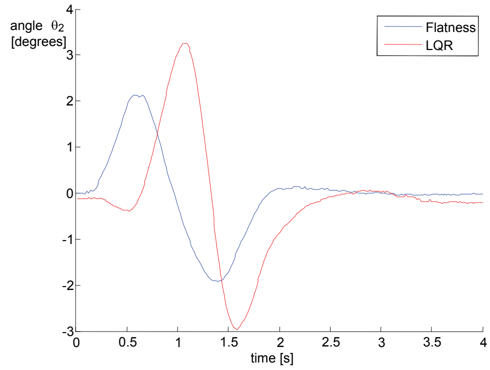

Fig. 4. Trajectory tracking of θ_{2r} (dashed) using flatness (blue) and LQR (red)

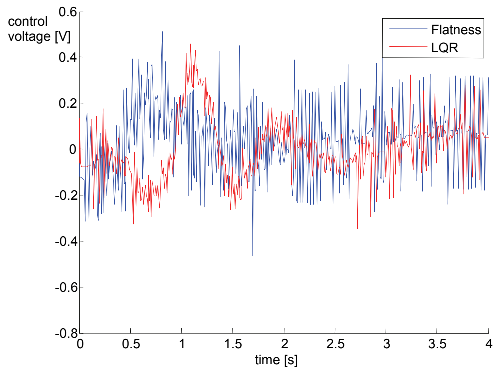

Fig. 5. Voltage variations at DC motor terminals

For the case of flatness control, disk follows the given trajectory almost perfect, while for LQR it deviates significantly. Also, in the case of LQR, pendulum oscillations are larger. In the case of flatness algorithm voltage variations are more pronounced compared to the LQR. LQR algorithm performs optimization of energy consumption, while the flatness control achieves accurate and fast trajectory tracking.

6. CONCLUSION

This paper shows the comparative analysis of two algorithms for Furuta pendulum control. LQR is a classic method for linear control systems and can be used for this type of "underactuated" system. However, LQR algorithm shows significant shortcomings in reference trajectory tracking. Flatness control, as a technique for non-linear systems control, overcomes these deficiencies. The experimental results confirm the characteristics of both algorithms. Further direction of research will refer to the comparison of flatness method with other methods for nonlinear systems, and testing the robustness, parameter sensitivity and tracking error convergence.

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