THE ANALYSIS OF THE EFFICIENCY OF POLYA RATIONAL PARAMETRIC INTERPOLATION KERNEL IN ESTIMATING THE FUNDAMENTAL FREQUENCY OF THE SPEECH SIGNAL

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Abstract: The paper describes Polya frequency functions, and the construction of Polya rational parametric interpolation kernel. The paper also provides the results of the estimation of fundamental frequency of speech signal obtained by applying the convolution interpolation algorithm with an implemented Polya rational parametric kernel. The analysis of the estimation of fundamental frequency was performed on a speech signal superimposed by AWGN when SNR=0-50 dB. In the processing of a signal in time domain some common window functions were applied. Then, MSE was used as a measurement of the quality of estimation to determine the optimum values of Polya interpolation kernel parameter. A comparative analysis was performed with the results of the estimate of the fundamental frequency by using the Keys one-parameter, quadratic and Polya quasi-rational kernel.

1. INTRODUCTION

In many scientific disciplines there is a need to analyze scattered data. The scattered data represent a set made of n irregularly distributed points $P_i\,(x_i,y_i)$, i=1, 2, ..., n in a xOy plane. Compared to the regular grid, the points P_i are irregularly distributed, that is, they are scattered inside the cells of the regular grid. By the process of grid regularization, called griding, all P_i points are allocated in the vertices of the regular grid. This allows data

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processing by using algorithms developed for the data which are shown in regular grids. This problem is solved by using interpolation algorithm or approximation algorithm. The analysis of scattered data is also performed by radial basis functions (*RBF*) [1]-[3]. These functions are used intensively for the numerical solving of partial differential equations, neural networks etc. [4], [5]. The invariability feature of *RBFs* in relation to translation, rotation and reflection makes them suitable for implementation in digital image processing. In their papers [6]-[8], Bochner and Schoenberg have shown important results in the field of *RBF* study. Based on their theorems, they derived the equations for interpolation kernels which are suitable for the interpolation of the scattered data. Different interpolation kernels offer different precision and efficiency of the interpolation algorithms. Apart from that, different kernels have different numerical complexity, as well as different time of processing. Paper [9] shows parametric interpolation kernels derived by using Polya frequency functions. When using parametric kernels, it is possible to influence the precision of the interpolation function by changing the values of kernel parameter, in other words, it is possible to adapt the kernel to a problem according to some criterion [10]-[14].

To estimate the fundamental frequency (f_0) the time and spectral domain [13]-[18] can be used. For working in the spectral domain, first the Discrete Fourier Transform (DFT) needs to be performed over the discrete signal. As a result, DFT provides the approximation of the signal spectrum. Namely, DFT is calculated on frequencies f_k for k=0, 1,…, *NDFT*-1, where *NDFT* is the length of DFT. Considering its energy, the fundamental frequency is the largest component in the spectral domain. However, in the case when the fundamental frequency differs from the calculating frequency of DFT, the estimation of spectral components will be wrong because of the leakage effect of the spectrum. It is possible to improve the precision of the estimation of the f_0 by using interpolation. However, interpolation function can be of an impractically high order, which as a consequence provides a more complex numerical algorithm and, therefore, a longer calculating time. To make a compromise, the solution is to apply the convolution interpolation with polynomial kernels of a lower order [10]-[12]. In his paper [10], Keys suggests parametric convolution kernel of the third order. By using the Taylor extension along with minimizing the interpolation error, Keys suggested the optimum value of the parameter (α=-0.5). The interpolation kernel from [10] used with the suggested parameter is suitable for image processing. When f_0 was estimated by using the Keys parametric kernel, the suggested optimal kernel parameter, α, did not show satisfactory results. Therefore, it was necessary to determine the optimal parameter values for the Keys kernel to estimate the f_0 [13], [14].

This paper analyzes the use of Polya rational parametric interpolation kernel for the estimation of the fundamental frequency of the sinusoidal and speech signal in the spectral domain. The authors of this paper have formed a Polya rational parametric interpolation kernel based on the Polya kernel from the paper [9]. The analysis of the precision of the estimate of the f_0 was performed for the sinusoidal and speech signal when they were superimposed by Additive White Gaussian Noise (AWGN) in the following range: *SNR*= 0-50 dB. The precision of estimating the fundamental frequency was measured by using the Mean Square Error (*MSE*). A greater precision was obtained by the processing of a time discrete signal by using some of the standard window functions (Hann, Hamming, …) and by choosing the optimal parameter of the convolution kernel. Based on the comparative analysis of the results of the application of Keys one-parameter [10], quadratic [14] and Polya quasi-rational kernel [18] the efficiency of the suggested Polya rational parametric interpolation kernel used in the estimation of the fundamental frequency of the speech signal was estimated.

Further organization of this paper is as follows: Section 2 describes Polya frequency functions. Section 3 shows the Polya rational parametric interpolation kernel. Section 4 shows the experiment, the received results and their analysis. Section 5 presents the conclusion.

2. POLYA FREQUENCY FUNCTIONS

This paper analyzes the efficiency of the convolution interpolation with Polya interpolation kernel. In the process of creating a Polya kernel the starting points are: a) positive definite and b) radial functions.

A. Positive definite functions

Definition 1. A continuous complex valued function $f : R^d \to C$ is a positive definite function if

$$\sum_{j=1}^{N}\sum_{k=1}^{N} c_j \overline{c_k} f(x_j - x_k) \geq 0, \tag{1}$$

for an arbitrary choice $x_1,...x_N \in R^d, c_1,...c_N \in C$. The function $f(x)$ is called strictly a positive definite on R^d, if the quadratic form in (1) is greater than 0 for $c_1,...,c_N \in C/\{0\}$.

One of the most important results of positive definite functions and their characterization in terms of the Fourier Transform on set *R*, was presented by Bochner 1932. and in 1933. on set R^d [6].

Theorem 1 (Bochner). A complex function $f(x)$ is a positive definite function on R^d if and only if $f(x)$ is the Fourier Transform of the finite nonnegative Borel measure μ on R^d, that is, if the following applies $f(x) = \int_{R^d} e^{-ixy} d\mu(y)$. In addition, if μ is a non-negative finite Borel measure on R^d whose carrier is not the set of Lebegsue measure zero, then $f(x)$ is strictly a positive definite.

The proof of this theorem is provided in [6].

B. Radial functions

Definition 2. Function $f(x)$ is radial if $f(x) = F(\|x\|)$, where $\|x\|$ is the Euklid norm on R^d.

In paper [7] Schoenberg provides a characterization of positive definite radial functions.

Theorem 2 (Schoenberg). A continuous function $f(x) = F(\|x\|)$ is positive definite and radial on R^d for every d=1,2,... if and only if it can be presented in the form:

$$F(r) = \int_0^\infty e^{-r^2 t^2} d\mu(t) , \tag{2}$$

where μ is the finite nonnegative Borel measure on $[0, \infty)$.

The proof of this theorem can be found in [7].

C. Polya frequency functions

Definition 3. A nonnegative measurable function, $\Lambda(x)$, which on R complies with the condition, $0 < \int_R \Lambda(x) dx < \infty$, is called a Polya frequency function if it complies with the following condition: for each two sets of strictly ascending numbers

$$x_1 < x_2 < \cdots x_n \quad y_1 < y_2 < \cdots y_n \; n=1,2,... \tag{3}$$

the following condition is met:

$$\det\{\Lambda(x_i - y_j)\}_{1,n} \geq 0. \tag{4}$$

Schoenberg gives the necessary and sufficient conditions for integrable functions to be called Polya frequency functions.

Theorem 3 (Schoenberg). The two-sided Laplace transformation of Polya frequency function $\Lambda(x)$ converges in a vertical strip and can be written as follows:

$$\int_{-\infty}^{\infty} e^{-sx} \Lambda(x) dx = \frac{1}{\Psi(s)}, \tag{5}$$

where $\Psi(s)$ is an entire function in this form:

$$\Psi(s) = Ce^{-\gamma s^2 - \delta_0 s} \prod_{m=1}^{\infty} (1 + s\delta_m)^{-s\delta_m} , \tag{6}$$

$$C > 0,\ \gamma \geq 0,\ \delta_m \in R, \quad 0 < \gamma + \sum_{m=1}^{\infty} \delta_m^2 < \infty. \tag{7}$$

Besides, when $\gamma > 0$, the function $\Lambda(x)$>0 is of class $C^{\infty}(R)$ and its derivatives $\Lambda^{(n)}(x)$ have only *n* simple real zeros for all *n* values.

The proof of this theorem is provided in [8].

An interesting consequence of this theorem is the existence of Polya frequency function $\Lambda(x)$ whose two-sided Laplace transformation is a quasi-rational function. (it can be written as a product of a rational and an entire function). Namely, by using $\delta_m = 0$ in the equation (6) when $m > M_0 \geq 1$ this follows:

$$\int_{-\infty}^{\infty} e^{-sx}\Lambda(x)dx = \frac{1}{C} e^{\gamma s^2 + \sum_{m=0}^{M_0}\delta_m s} \prod_{m=1}^{M_0} \frac{1}{1+s\delta_m}. \tag{8}$$

By using $s = i\omega$ in equation (5) this follows:

$$\int_{-\infty}^{\infty} e^{-i\omega x}\Lambda(x)dx = \frac{1}{\Psi(i\omega)}. \tag{9}$$

On the other hand, by using this change in equation (8) this follows:

$$\int_{-\infty}^{\infty} e^{-i\omega x}\Lambda(x)dx = \frac{1}{C} e^{\gamma(i\omega)^2 + \sum_{m=0}^{M_0}\delta_m i\omega} \prod_{m=1}^{M_0} \frac{1}{1+i\omega\delta_m}. \tag{10}$$

By applying C =1, $M_0 = 2$, $\gamma = \delta_0 = 0$, $0 < c = \delta_1 = -\delta_2$ in (10) the equation for rational Polya kernel is obtained:

$$\int_{-\infty}^{\infty} e^{-i\omega x}\Lambda(x)dx = \frac{1}{1+c^2\omega^2} = h(\omega), \tag{11}$$

where $\Lambda(x)$ is a Polya frequency function:

$$\Lambda(x) = \frac{1}{2c} e^{\left(-\frac{|x|}{c}\right)}. \tag{12}$$

By using C =1, $M_0 = 1$, $\gamma = 0$, $0 < c = \delta_1 = -\delta_0$ in (10) the equation for Polya quasi-rational interpolation kernel is obtained:

$$\int_{-\infty}^{\infty} e^{-i\omega x} \Lambda(x) dx = \frac{1}{1 + i\omega c} = h(\omega), \tag{13}$$

with Polya frequency function:

$$\Lambda(x) = \frac{1}{c} e^{\left(-\frac{x}{c}\right)} \psi(x), \tag{14}$$

where $\psi(x)$ is the Heviside function.

3. POLYA RATIONAL PARAMETRIC INTERPOLATION KERNEL

A. Kernel

By using the analogy with Polya frequency function, that is, with its Fourier transform (11) the parametric interpolation kernel was constructed:

$$r(f) = \begin{cases} 1/(1 + \alpha^2 |f|^2), & k - 1 \le |f| \le k,\ k = 1,2...,L/2 \\ 0 & |f| > L/2 \end{cases}, \tag{15}$$

where α is a kernel parameter, and L kernel length. It is possible to adjust this parameter so that the characteristics of the kernel can adjust to the corresponding problem, in accordance with a criterion. Interpolation kernel (15) does not meet the condition $r(f_k) = 0$, which as a consequence leads to the following: the interpolated function cannot pass through the knots. Therefore, a function that is defined in this way represents the approximation of the function.

B. Algorithm for determining the interpolation kernel parameter

This paper analyzes the problem of estimating the fundamental frequency of the signal by an analysis in the spectral domain. Therefore, the parameter α will be chosen in such a way to minimize the error of estimating the fundamental frequency in the spectral domain. The algorithm for determining the parameter α of the interpolation kernel r is set up based on the following steps:

Input: Test signal $s(n)$, sequence length N, real fundamental frequency f_0, interpolation kernel r, NDFT - length DFT, SNR.
Output: Kernel parameter α_{opt}.

Step 1: Modification by using the window function w of N length:

$$s_w = s \cdot w. \tag{16}$$

Step 2: Through the implementation of the discrete Fourier transform the spectrum X is calculated:

$$X=\mathrm{DFT}(s_w, NDFT). \tag{17}$$

In this equation, NDFT stands for the length of DFT.

Step 3: Peaking method is used to obtain the position of the spectral component with the highest amplitude:

$$k_{max}=peak\_picking(X). \tag{18}$$

Step 4: By using the convolution interpolation in the neighbourhood of k_{max} the reconstructed function is calculated $X_r(f)$.

The reconstructed function is:

$$X_r(f)=\sum_{i=-1}^{2} p_i r(f-i), \tag{19}$$

where p_i=$X(i)$, $r(f)$ is the interpolation kernel, and $k \leq f \leq k+1$.

Step 5: To determine the position of the maximum of the reconstructed function $X_r(f)$, to equate the first derivative with zero and to estimate the fundamental frequency f_e.

$$\frac{d(X_r(f))}{df}=0 \;\Rightarrow\; f_e=f\ . \tag{20}$$

Step 6: Calculating the *MSE* between the estimated f_e and the real f_0 fundamental frequency depending on the α parameter,

$$MSE=\overline{(f_o-f_e)^2}\ . \tag{21}$$

Step 7: Locating the minimum *MSE* and calculating the optimal value of the kernel parameter α_{opt}.

C. Test signal

Algorithm of the estimate of f_0 will be implemented on:

a) simulated sinusoidal test signal and

b) real speech test signal.

Simulated sinusoidal signal for the testing of the interpolation algorithm is defined in paper [17]:

$$s(t)=\sum_{i=1}^{K}\sum_{g=0}^{M}a_i \sin\left(2\pi i\left(f_0+g\frac{f_0}{KM}\right)t+\theta_i\right), \tag{22}$$

where f_0 is the fundamental frequency, a_i and θ_i amplitude and phase of the i-th harmonic respectfully, K the number of harmonics, and M the number of points between two samples.

The speech test signal is obtained by the recording of speech in a real acoustic environment [13].

4. EXPERIMENTAL RESULTS AND ANALYSIS

A. Experiment

The estimating of the optimum parameter of Polya rational interpolation kernel as well as the choice of the window function is realized by the implementation of algorithms for the estimation of parameters (described in section 3. B) on the test signal.

In this experiment, the parameters of the sinusoidal and speech test signal are as follows: f_0=125-140 Hz, sampling frequency f_s=8 kHz, block length N=256 (32 *ms*), K=10, M=100. Then the analysis of the efficiency of the estimate was performed when AWGN was superimposed over the test signals. The analysis was performed for the case when the values were SNR= {0, 10, 20, 30, 50} dB. The implemented windows were: Hamming, Hann, Blackman, Rectangular, Kaiser and Triangular.

B. Results

By using the Polya rational parameter kernel on the sinusoidal and speech test signals with the implementation of window functions, the results for MSE_{min} and α_{opt} were obtained. They are shown in table I and in figures 1-6. In order to compare the results, table II also provides the results obtained by the implementation of Keys one-parameter quadratic interpolation kernel in [13], and table III shows the results of the quadratic interpolation kernel in [14] and Polya quasi-rational kernel on the sinusoidal signal [18]. Table IV shows the MSE values for different values of SNR and different window functions for the sinusoidal test signal. Table V offers the MSE values for different values of SNR and different window functions for the speech test signal.

Table I
Minimum MSE and α_{opt} for the application of Polya rational kernel for a sinusoidal and speech test signal.

Window	**Sine test signal**		**Speech test signal**	
	α_{opt}	MSE_{min}	α_{opt}	MSE_{min}
Hamming	-0.400	0.0058	-0.380	0.0353
Hann	-0.400	0.0133	-0.400	0.0447
Blackman	-0.700	0.0300	-0.650	0.0607
Rectang.	-0.050	0.6712	-0.250	0.2521
Kaiser	-0.600	0.0138	-0.600	0.0421
Triangular	-0.400	0.0024	-0.400	0.0244

Table II
Minimum MSE and α_{opt} for the application of the Keys kernel for a sinusoidal and speech test signal.

Window	**Sine test signal**		**Speech test signal**	
	α_{opt}	MSE_{min}	α_{opt}	MSE_{min}
Hamming	-1.005	0.023	-0.995	0.0310
Hann	-0.885	0.004	-0.880	0.0349
Blackman	-1.801	0.001	-0.800	0.0358
Rectang.	-2.61	0.515	-2.400	0.4323
Kaiser	-1.125	0.020	-1.080	0.0339
Triangular	-1.028	0.0028	-1.028	0.0277

Table III
Minimum MSE and α_{opt} for the application of Polya quasi-rational and quadratic interpolation kernel for a sinusoidal test signal.

Window	**Polya quasi-rational kernel**		**Quadratic kernel**	
	α_{opt}	MSE_{min}	α_{opt}	MSE_{min}
Hamming	-0.45	0.0068	-1.125	0.8727
Hann	-0.45	0.0138	-1.100	0.899
Blackman	-0.70	0.0300	-0.915	0.6014
Rectang.	-0.06	0.6717	-0.010	0.0726
Kaiser	-0.70	0.0155	-1.065	0.963
Triangular	-0.45	0.0044	-1.140	1.0026

Table IV
MSE values depending on SNR and the window for the sinusoidal test signal.

Window	MSE				
	0 dB	**10 dB**	**20 dB**	**30 dB**	**50 dB**
Hamming	0.7559	0.1999	0.0372	0.0106	0.0058
Han	1. 4124	0.2313	0.0411	0.0150	0.0133
Blackman	1.7704	0.2178	0.0503	0.0343	0.0300
Rectangular	5.9290	1.9925	1.0097	0.7851	0.6712
Kaiser	0.1523	0.0645	0.0208	0.0131	0.0138
Triangular	0.5347	0.1678	0.0302	0.0089	0.0024

Table V
MSE values depending on SNR and the window for the speech test signal.

Window	MSE				
	0 dB	**10 dB**	**20 dB**	**30 dB**	**50 dB**
Hamming	0.7075	0.0626	0.0184	0.0251	0.0353
Han	0.9198	0.0648	0.0186	0.0263	0.0447
Blackman	1.5207	0.1448	0.0624	0.0609	0.0607
Rectangular	2.2466	0.3959	0.2601	0.2601	0.2521
Kaiser	0.5045	0.0573	0.0193	0.0338	0.0421
Triangular	0.6771	0.0541	0.0227	0.0254	0.0244

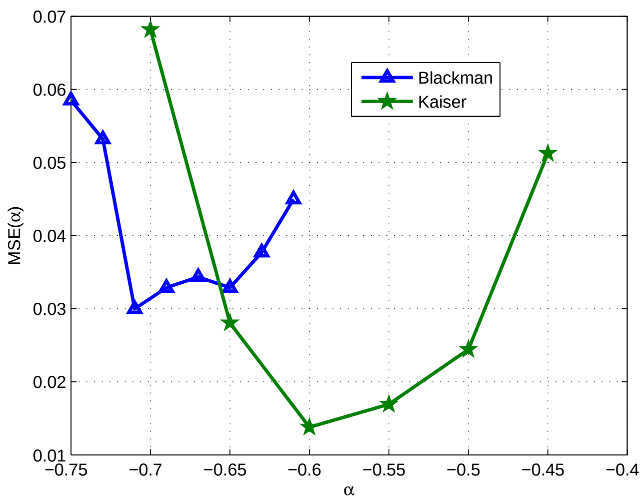

Fig. 1. The dependence of MSE on α for the application of the Blackman and Kaiser window in interpolation with a rational Polya kernel for a sinusoidal test signal.

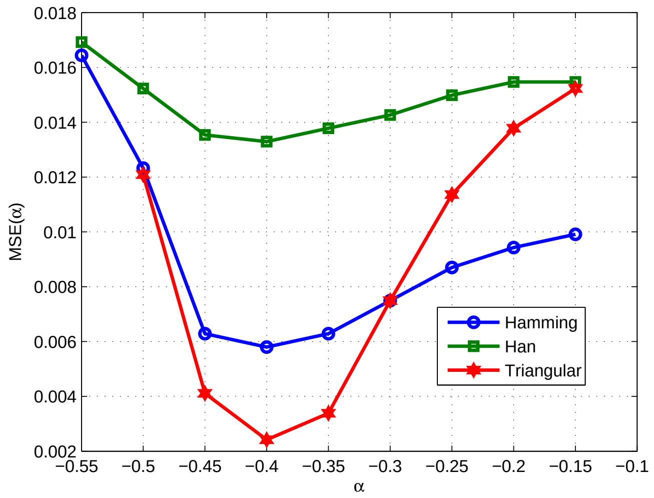

Fig. 2. The dependence of MSE on α for the application of Hamming, Hann, and Triangular window in interpolation with a rational Polya kernel for a sinusoidal test signal.

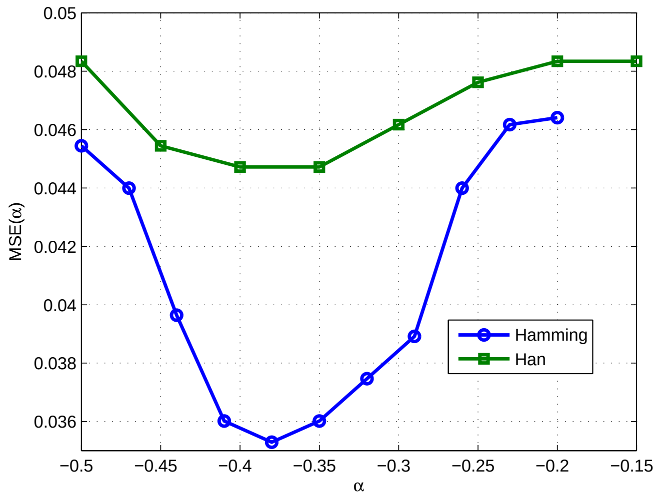

Fig. 3. The dependence of MSE on α for the application of Hamming and Hann window in interpolation with a rational Polya kernel for a speech test signal.

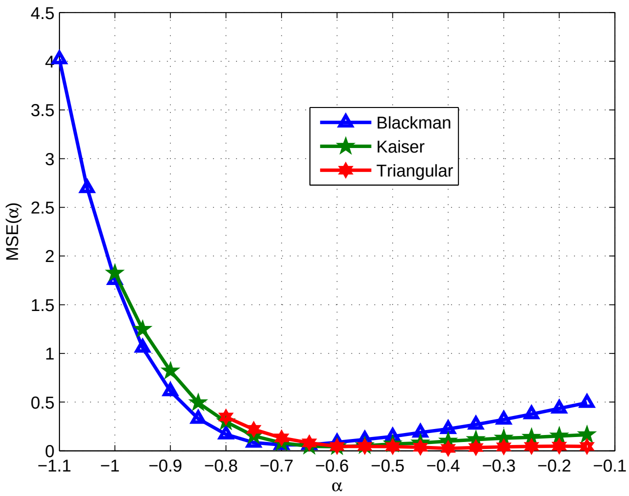

Fig. 4. The dependence of MSE on α for the application of Blackman, Kaiser and Triangular window in interpolation with a rational Polya kernel for a speech test signal.

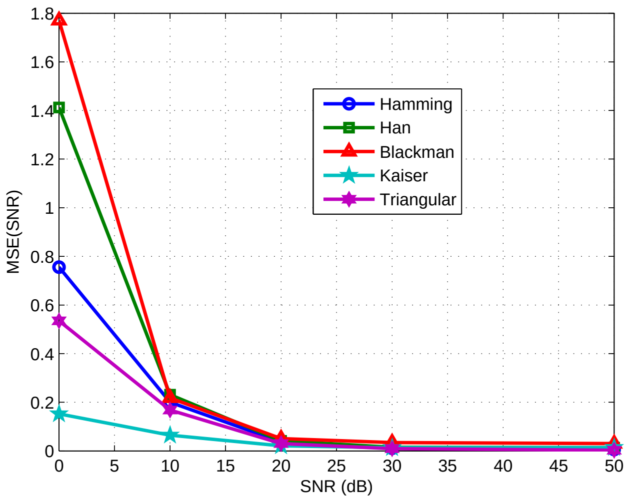

Fig. 5. The dependence of MSE on SNR in case of implementation of window functions in interpolation by using the rational Polya kernel for a sinusoidal test signal.

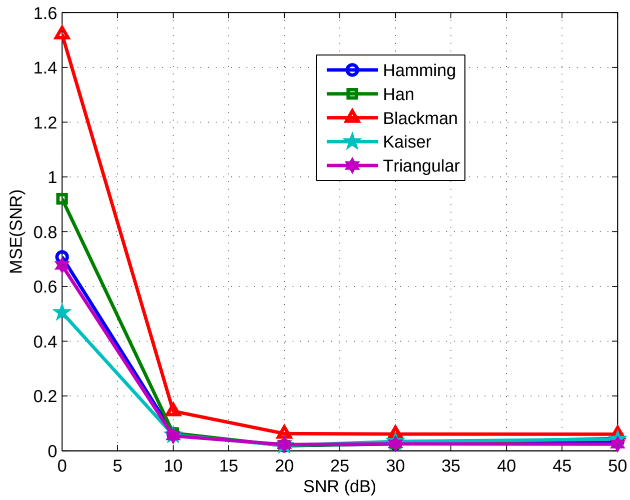

Fig. 6. The dependence of MSE on SNR in case of implementation of window functions in interpolation by using the rational Polya kernel for a speech test signal.

C. Analysis of the results

Based on the results above, it is concluded that:

a) in case of application of the rational Polya kernel on a sinusoidal signal the smallest error was calculated for the triangular window function. In comparison to other window functions, the triangular one has shown better results: a) 59% (Hamming), b) 82% (Han), c) 92% (Blackman), d) 83% (Kaiser) and e) 99% (Rectangular). The biggest error appeared when the Rectangular window function was used,

b) with a speech signal, when interpolation by a rational Polya kernel was used, the application of the Triangular window function provided the smallest error: a) 31% (Hamming), b) 45% (Han), c) 60% (Blackman), d) 42% (Kaiser) and e) 90% (Rectangular), while the application of the Rectangular window function provided the biggest error,

c) in case of the sinusoidal signal, compared to Keys one-parameter cubic convolution kernel [12] which gave the best results when Blackman window was applied, the Polya kernel showed a $MSE_{\min\_triang\_Polya\_sin} / MSE_{\min\_Black.\_Keys\_sin}$=0.0024/0.001=2.4 times bigger error,

d) the rational Polya kernel showed greater efficiency compared to the Keys kernel, on a speech signal when the Triangular window function was applied. After the comparison of the received results with the results for Keys one-parameter kernel [12] it was concluded that Polya kernel showed a $MSE_{min\_triang\_Keys\_sp.}/MSE_{min\_triang\_Polya\ \_sp.}$=0.0277/0.0244=1.14 times smaller error,

e) by comparing the results received by the application of quadratic interpolation kernel [13] on a sinusoidal signal where the smallest MSE was for the Rectangular window function, it was concluded that the rational Polya kernel has a $MSE_{\min\_rectang\_quadrat}$/ $MSE_{\min\_triang\_Polya}$=0.0726/0.0024=30.25 times smaller Mean Square Error,

f) compared to the quasi-rational Polya interpolation kernel, for which the best results were obtained by using the Triangular window function, the suggested kernel on a sinusoidal signal has shown a $MSE_{\min\_triang\_quasi\_Polya} / MSE_{\min\_triang\_Polya}$=0.0044/0.0024=1.83 times smaller error,

g) the estimate of the precision of sinusoidal compared to a speech test signal when interpolation by Polya rational kernel is used after the application of window functions is:

$MSE_{\min\_triangl\_Polya\_sp}/MSE_{\min\_triang\_Polya\_sin}$=0.0244/0.0024=10.16 times bigger,

h) when using the rational Polya interpolation with the application of the Rectangular window function the speech signal is estimated more precisely. The error is $MSE_{\min\_rectang\ Polya\_sin} / MSE_{\min\_rectang\_Polya\_sp}$=0.06712/0.2521=2.67 times smaller in the estimate of a speech signal,

j) as *SNR* increases *MSE* decreases. In the case of a sinusoidal signal, the estimate of precision when *SNR*=50 dB compared to *SNR*=0 dB (Triangular window) is $MSE_{\min\_triangl\_Polya\_sin\_0}/MSE_{\min\_triang\_Polya\_sin\_50}$=0.6771/0.0244=27.5 times bigger,

k) greater precision in a speech signal in case of the implementation of the Triangular window function is with the increase of *SNR* (the case when *SNR*=0 dB and *SNR*=50 dB) $MSE_{\min\_triangl\_Polya\_sin\_0}/MSE_{\min\_triang\_Polya\_sin\_50}$=0.5347/0.0024=222.79 times bigger.

5. CONCLUSION

This paper shows the results of the implementation of parametric rational Polya convolution kernel for estimating the fundamental frequency of the sinusoidal and speech signal. In order to minimize MSE, some window functions were implemented. It can be discerned that the best results, both with the sinusoidal and the speech signal, were obtained by the application of the Triangular window function. This kernel makes a more precise estimate of a sinusoidal signal, except in the case of implementation of the Rectangular window function, where greater precision was obtained for the speech signal. After comparing the obtained results with the results of the estimate of the fundamental frequency by using the quadratic convolution kernel in [14], Keys one-parametric kernel in paper [13] and quasi-rational Polya kernel in [18], it can be concluded that the estimate of fundamental frequency on a sinusoidal test signal by using the rational Polya kernel is 30.25 times more precise compared to the estimate in the case when the quadratic kernel was used, and 1.83 times compared to the estimate by using the quasi-rational Polya kernel, while it is 2.4 times smaller than the precision obtained by using the Keys one-parameter convolution kernel. In the case of the speech test signal, Polya rational kernel showed a 1.14 times greater precision compared to the estimate obtained by using the Keys kernel. Due to its small numerical complexity, the Polya rational kernel can be used for working in real time.

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